

# Holomorphic Morse inequalities for orbifolds

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## Abstract

We prove that Demailly's holomorphic Morse inequalities hold true for complex orbifolds by using a heat kernel method.

## 1 Introduction

Morse Theory investigates the topological information carried by Morse functions on a manifold and in particular their critical points. Let  $f$  be a Morse function on a compact manifold of real dimension  $n$ . We suppose that  $f$  has isolated critical points. Let  $m_j$ , ( $0 \leq j \leq n$ ) be the number of critical points of  $f$  of Morse index  $j$ , and let  $b_j$  be the Betti numbers of the manifold. Then the strong Morse inequalities states that for  $0 \leq q \leq n$ ,

$$\sum_{j=0}^q (-1)^{q-j} b_j \leq \sum_{j=0}^q (-1)^{q-j} m_j, \quad (1.1)$$

with equality if  $q = n$ . From (1.1), we get the weak Morse inequalities:

$$b_j \leq m_j \quad \text{for } 0 \leq j \leq n. \quad (1.2)$$

In his seminal paper [20], Witten gave an analytic proof of the Morse inequalities by analyzing the spectrum of the Schrödinger operator  $\Delta_t = \Delta + t^2 |df|^2 + tV$ , where  $t > 0$  is a real parameter and  $V$  an operator of order 0. For  $t \rightarrow +\infty$ , Witten shows that the spectrum of  $\Delta_t$  approaches in some sense the spectrum of a sum of harmonic oscillators attached to the critical point of  $f$ .

In [5], Demailly established analogous asymptotic Morse inequalities for the Dolbeault cohomology associated with high tensor powers  $L^p := L^{\otimes p}$  of a (smooth) holomorphic Hermitian line bundle  $(L, h^L)$  over a (smooth) compact complex manifold  $(M, J)$  of dimension  $n$ . The inequalities of Demailly give asymptotic bounds on the Morse sums of the Betti numbers of  $\bar{\partial}$  on  $L^p$  in terms of certain integrals of the Chern curvature  $R^L$  of  $(L, h^L)$ . More precisely, we define  $\dot{R}^L \in \text{End}(T^{(1,0)}M)$  by  $g^{TM}(\dot{R}^L u, \bar{v}) = R^L(u, \bar{v})$  for  $u, v \in T^{(1,0)}M$ , where  $g^{TM}$  is any  $J$ -invariant Riemannian metric on  $TM$ . We denote by  $M(q)$  the set of points where  $\dot{R}^L$  is non-degenerate and exactly  $q$  negative eigenvalues, and we set  $M(\leq q) = \cup_{j \leq q} M(j)$ . Let  $n = \dim_{\mathbb{C}} M$ , then we have for  $0 \leq q \leq n$

$$\sum_{j=0}^q (-1)^{q-j} \dim H^j(M, L^p) \leq \frac{p^n}{n!} \int_{M(\leq q)} (-1)^q \left( \frac{\sqrt{-1}}{2\pi} R^L \right)^n + o(p^n), \quad (1.3)$$

with equality if  $q = n$ . Here  $H^j(M, L^p)$  denotes the Dolbeault cohomology in bidegree  $(0, j)$ , which is also the  $j$ -th group of cohomology of the sheaf of holomorphic sections of  $L^p$ . Note that  $M(q)$  and  $M(\leq q)$  are open subsets of  $M$  and do not depend on the metric  $g^{TM}$ .

These inequalities have found numerous applications. In particular, Demailly used them in [5] to find new geometric characterizations of Moishezon spaces, which improve Siu's solution in [17, 18] of the Grauert-Riemenschneider conjecture [9]. Another notable application of the holomorphic Morse inequalities is the proof of the effective Matsusaka theorem by Siu [19, 7]. Recently, Demailly used these inequalities in [8] to prove a significant step of a generalized version of the Green-Griffiths-Lang conjecture.

To prove these inequalities, the key remark of Demailly was that in the formula for the Kodaira Laplacian  $\square_p$  associated with  $L^p$ , the metric of  $L$  plays formally the role of the Morse function in the paper Witten [20], and that the parameter  $p$  plays the role of the parameter  $t$ . Then the Hessian of the Morse function becomes the curvature of the bundle. The proof of Demailly was based on the study of the semi-classical behavior as  $p \rightarrow +\infty$  of the spectral counting functions of  $\square_p$ . Subsequently, Bismut gave an other proof of the holomorphic Morse inequalities in [2] by adapting his heat kernel proof of the Morse inequality [1]. The key point is that we can compare the left hand side of (1.3) with the alternate trace of the heat kernel acting on forms of degree  $\leq q$ , i.e.,

$$\sum_{j=0}^q (-1)^{q-j} \dim H^j(M, L^p) \leq \sum_{j=0}^q (-1)^{q-j} \operatorname{Tr} \left[ e^{-\frac{q}{p} \square_p} |_{\Omega^{0,j}(M, L^p)} \right], \quad (1.4)$$

with equality if  $q = n$ . Then, Bismut obtained the holomorphic Morse inequalities by showing the convergence of the heat kernel thanks to probability theory. Demailly [6] and Bouche [4] gave an analytic approach of this result. In [15], Ma and Marinescu gave a new proof of this convergence, replacing the probabilistic arguments of Bismut [2] by arguments inspired by the analytic localization techniques of Bismut-Lebeau [3, Chap. 11].

When the bundle  $L$  is positive, (1.3) is a consequence of the Hirzebruch-Riemann-Roch theorem and of the Kodaira vanishing theorem, and reduces to

$$\dim H^0(M, L^p) = \frac{p^n}{n!} \int_M \left( \frac{\sqrt{-1}}{2\pi} R^L \right)^n + o(p^n). \quad (1.5)$$

In this case, a local estimate can be obtained by the study of the asymptotic of the Bergman kernel (the kernel of the orthogonal projection from  $\mathcal{C}^\infty(M, L^p)$  onto  $H^0(M, L^p)$ ) when  $p \rightarrow +\infty$ . We refer to [15] and the reference therein for the study of the Bergman kernel.

It is a natural question to know whether we can prove a version of Demailly's holomorphic Morse inequalities when  $M$  is a complex orbifold and  $L$  is an orbifold bundle. Applying the result of [10, 11], one can prove that such inequalities hold if  $M$  is the quotient of a CR manifold by a transversal  $\mathbb{S}^1$ -action. In this paper, we prove that Demailly's inequalities hold for high tensor power of an orbifold line bundle, twisted by another orbifold bundle, on a general complex orbifolds.

We now give more details about our result.

Let  $M$  be a compact complex orbifold of dimension  $n$ . We denote the complex structure of  $M$  by  $J$ . Let  $(L, h^L)$  a Hermitian holomorphic orbifold line bundle on  $M$  and let  $(E, h^E)$  be a Hermitian holomorphic orbifold vector bundle on  $M$ . As we will see in Section 2.3, we may assume without loss of generality that  $L$  and  $E$  are proper. We denote by  $R^L$  the Chern curvature of  $(L, h^L)$ . We refer to Section 2 for the background concerning orbifold, but for this introduction, let us just say that every object on orbifold can be seen locally as being the quotient of an object on a non-singular manifold which is invariant by a finite group, and that we keep the same notation for both these objects.

We define  $\hat{R}^L$ ,  $M(q)$  and  $M(\leq q)$  exactly as in the non-singular case above.

Let  $H^\bullet(M, L^p \otimes E)$  be the orbifold Dolbeault cohomology. Then our holomorphic Morse inequalities for orbifolds have the same statement as the regular ones :

**Theorem 1.1.** *As  $p \rightarrow +\infty$ , the following strong Morse inequalities hold for  $q \in \{1, \dots, n\}$*

$$\sum_{j=0}^q (-1)^{q-j} \dim H^j(M, L^p \otimes E) \leq \text{rk}(E) \frac{p^n}{n!} \int_{M(\leq q)} (-1)^q \left( \frac{\sqrt{-1}}{2\pi} R^L \right)^n + o(p^n), \quad (1.6)$$

with equality for  $q = n$ .

In particular, we get the weak Morse inequalities

$$\dim H^q(M, L^p \otimes E) \leq \text{rk}(E) \frac{p^n}{n!} \int_{M(q)} (-1)^q \left( \frac{\sqrt{-1}}{2\pi} R^L \right)^n + o(p^n). \quad (1.7)$$

Of course, if the orbifold  $M$  is a non-singular manifold, and if the orbifold bundles  $L$  and  $E$  are usual bundles, then Theorem 1.1 gives back the inequalities of Demailly.

Let us now give the main steps of our proof. We draw our inspiration from the heat kernel method of [2] (see also [15, Sects. 1.6 and 1.7]), but the main difficulty when compared to this paper is that the singularities of  $M$  make it impossible to have a uniform asymptotics for the heat kernel (see Remark 1.6), and we thus have to work further near the singularities. In particular we have to use the off-diagonal development of the heat kernel.

Note that in the case where  $L$  is positive, similar difficulties arise in the study of the Bergman kernel, and the asymptotic results concerning the heat kernel given below have parallel results for the Bergman kernel on orbifolds [15].

Let  $g^{TM}$  be a Riemannian metric on  $TM$  which is compatible with  $J$ . Let

$$\square_p := \bar{\partial}^{L^p \otimes E} \bar{\partial}^{L^p \otimes E, *} + \bar{\partial}^{L^p \otimes E, *} \bar{\partial}^{L^p \otimes E} \quad (1.8)$$

be the Kodaira Laplacian acting on  $\Omega^{0,\bullet}(M, L^p \otimes E)$  associated with  $L^p \otimes E$ ,  $h^L$ ,  $h^E$  and  $g^{TM}$  (see the beginning of Section 3 for details). Note that the operator  $\square_p$  preserves the  $\mathbb{Z}$ -grading. We denote by  $\text{Tr}_q[e^{-\frac{u}{p}\square_p}]$  the trace of  $e^{-\frac{u}{p}\square_p}$  acting on  $\Omega^{0,q}(M, L^p \otimes E)$ . We then have an analogue of (1.4):

**Theorem 1.2.** *For any  $u > 0$ ,  $p \in \mathbb{N}^*$  and  $0 \leq q \leq n$ , we have*

$$\sum_{j=0}^q (-1)^{q-j} \dim H^j(M, L^p \otimes E) \leq \sum_{j=0}^q (-1)^{q-j} \text{Tr}_j \left[ e^{-\frac{u}{p}\square_p} \right], \quad (1.9)$$

with equality for  $q = n$ .

From this result, we will prove our inequalities (1.6) by studying the asymptotics of the heat kernel. We begin by its behavior away from the singularities of  $M$ .

Let  $\{w_j\}_{j=1}^n$  be a local orthonormal frame of  $T^{(1,0)}M$  (with the metric induced by  $g^{TM}$ ) with dual frame  $\{w^j\}_{j=1}^n$ . Set

$$\omega_d = - \sum_{k,\ell} R^L(w_k, \bar{w}_\ell) \bar{w}^\ell \wedge i_{\bar{w}_k} \in \text{End}(\Lambda^{0,\bullet}(T^*M)). \quad (1.10)$$

For compactness, in the sequel we will write

$$\mathcal{L}im_u(x) = \frac{1}{(2\pi)^n} \frac{\det(\dot{R}_x^L) e^{u\omega_{d,x}}}{\det(1 - \exp(-u\dot{R}_x^L))} \otimes \text{Id}_{E_x}, \quad (1.11)$$

with the convention that if an eigenvalue of  $\dot{R}_x^L$  is zero, then its contribution to the term  $\frac{\det(\dot{R}_x^L)}{\det(1 - \exp(-u\dot{R}_x^L))}$  is  $\frac{1}{u}$ .

Let  $M_{reg}$  be the regular part of  $M$  (see Section 2) and let  $M_{sing} = M \setminus M_{reg}$  be the singular part of  $M$ .

**Theorem 1.3.** *For  $K \subset M_{reg}$  compact,  $u > 0$  and  $\ell \in \mathbb{N}$ , there exists  $C > 0$  such that for any  $x \in K$ , we have as  $p \rightarrow +\infty$*

$$\left| e^{-\frac{u}{p}\square_p}(x, x) - \mathcal{L}im_u(x) \right|_{\mathcal{C}^\ell} \leq Cp^{-1/2}. \quad (1.12)$$

Here,  $|\cdot|_{\mathcal{C}^\ell}$  denotes the  $\mathcal{C}^\ell$ -norm.

We now turn to the asymptotic behavior of the heat kernel near the singularities. This is the main technical innovation of this paper.

Let  $\nabla^L$  and  $\nabla^E$  be the Chern connections of  $(L, h^L)$  and  $(E, h^E)$ , i.e., the unique connections preserving both the holomorphic and Hermitian structure.

Let  $x_0 \in M_{sing}$ , from Section 2 (and in particular Lemma 2.4), we know that we can identify an open neighborhood  $U_{x_0}$  of  $x_0$  to  $\tilde{U}_{x_0}/G_{x_0}$ , where  $\tilde{U}_{x_0} \subset \mathbb{C}^n$  is an open neighborhood of 0 on which the finite group  $G_{x_0}$  acts linearly and effectively. In this chart,  $x_0$  correspond to the class  $[0]$ . Then the metric  $g^{TM}$  induces a  $G_{x_0}$ -invariant metric on  $\tilde{U}_{x_0}$ .

Let  $\tilde{U}_{x_0}^g$  be the fixed point-set of  $g \in G_{x_0}$ , and let  $\tilde{N}_{x_0, g}$  be the normal bundle of  $\tilde{U}_{x_0}^g$  in  $\tilde{U}_{x_0}$ . For each  $g \in G_{x_0}$ , the exponential map  $Y \in (\tilde{N}_{x_0, g})_{\tilde{x}} \mapsto \exp_{\tilde{x}}^{\tilde{U}_{x_0}}(Y)$  identifies a neighborhood of  $\tilde{U}_{x_0}$  to  $\tilde{W}_{x_0, g} = \{Y \in \tilde{N}_{x_0, g} : |Y| \leq \varepsilon\}$ . We identify  $L|_{\tilde{W}_{x_0, g}}$  and  $E|_{\tilde{W}_{x_0, g}}$  to  $L|_{\tilde{U}_{x_0}^g}$  and  $E|_{\tilde{U}_{x_0}^g}$  by using the parallel transport (with respect to  $\nabla^L$  and  $\nabla^E$ ) along the above exponential map. Then the action of  $g$  on  $L|_{\tilde{W}_{x_0, g}}$  is the multiplication by  $e^{i\theta_g}$ , and  $\theta_g$  is locally constant on  $\tilde{U}_{x_0}^g$ . Likewise, the action of  $g$  on  $E|_{\tilde{W}_{x_0, g}}$  is given by  $g^E \in \mathcal{C}^\infty(\tilde{U}_{x_0}^g, \text{End}(E))$ , and  $g^E$  is parallel with respect to  $\nabla^E$ .

If  $\tilde{Z} \in \tilde{W}_{x_0, g}$ , we write  $\tilde{Z} = (\tilde{Z}_{1, g}, \tilde{Z}_{2, g})$  with  $\tilde{Z}_{1, g} \in \tilde{U}_{x_0}^g$  and  $\tilde{Z}_{2, g} \in N_{x_0, g}$ .

On  $\tilde{U}_{x_0}$ , we have two metrics: the first is the  $G$ -invariant lift  $\widehat{g^{TM}}$  of  $g^{TM}|_{U_{x_0}}$  and the second is the constant metric  $(\widehat{g^{TM}})_{\tilde{Z}=0}$ . Let  $dv_{\tilde{M}}$  and  $dv_{\tilde{T}\tilde{M}}$  be the associated volume form, and let  $\tilde{\kappa}$  be the smooth positive function defined by

$$dv_{\tilde{M}}(\tilde{Z}) = \tilde{\kappa}(\tilde{Z})dv_{\tilde{T}\tilde{M}}(\tilde{Z}), \quad (1.13)$$

with  $\tilde{\kappa}(0) = 1$ .

For  $x \in U_{x_0}$ , the  $G$ -invariant lift of  $\dot{R}_x^L$  acts on  $T^{1,0}\tilde{U}_{x_0}$ , and we extend it to  $T\tilde{U}_{x_0} \otimes \mathbb{C} = T^{1,0}\tilde{U}_{x_0} \oplus T^{0,1}\tilde{U}_{x_0}$  by setting  $\dot{R}_x^L \tilde{v} = -\overline{\dot{R}_x^L \tilde{v}}$ . We then define

$$\mathcal{E}_{g, x}(u, \tilde{Z}) = \exp \left\{ - \left\langle \frac{\dot{R}_x^L/2}{\text{th}(u\dot{R}_x^L/2)} \tilde{Z}, \tilde{Z} \right\rangle + \left\langle \frac{\dot{R}_x^L/2}{\text{sh}(u\dot{R}_x^L/2)} e^{u\dot{R}_x^L/2} g^{-1} \tilde{Z}, \tilde{Z} \right\rangle \right\}. \quad (1.14)$$

Here again, we use the convention that if an eigenvalue of  $\dot{R}_x^L$  is zero, then the contribution of the associated eigenspace to  $\mathcal{E}_{g, x}(u, \tilde{Z})$  is of the form  $(u, \tilde{v}) \mapsto e^{-\frac{1}{2u}|g^{-1}\tilde{v}-\tilde{v}|^2}$ .

**Theorem 1.4.** *On  $\tilde{U}_{x_0}$ , for  $u > 0$  and  $\ell \in \mathbb{N}$ , there exist  $c, C > 0$  and  $N \in \mathbb{N}$  such that for any  $|\tilde{Z}| < \varepsilon/2$ , as  $p \rightarrow +\infty$*

$$\left| p^{-n} e^{-\frac{u}{p} \square_p}(\tilde{Z}, \tilde{Z}) - \mathcal{L}im_u(\tilde{Z}) - \sum_{\substack{g \in G_x \\ g \neq 1}} e^{ip\theta_g} g^E(\tilde{Z}_{1,g}) \kappa_{\tilde{Z}_{1,g}}^{-1}(\tilde{Z}_{2,g}) \mathcal{L}im_u(\tilde{Z}_{1,g}) \mathcal{E}_{g, \tilde{Z}_{1,g}}(u, \sqrt{p} \tilde{Z}_{2,g}) \right|_{\mathcal{C}^\ell} \leq Cp^{-1/2} + Cp^{\frac{\ell-1}{2}} (1 + \sqrt{p} d(Z, M_{sing}))^N e^{-cp d(Z, M_{sing})^2}. \quad (1.15)$$

*Remark 1.5.* The term  $\mathcal{L}im_u(\tilde{Z}_{1,g}) \mathcal{E}_{g, \tilde{Z}_{1,g}}(u, \sqrt{p} \tilde{Z}_{2,g})$  appearing in (1.15) can be seen as the heat kernel at the time  $u$  of some explicit harmonic oscillator (depending on  $\tilde{Z}_{1,g}$ ) on  $\mathbb{R}^{2n}$ , evaluated at  $(\sqrt{p}g^{-1}\tilde{Z}_{2,g}, \sqrt{p}\tilde{Z}_{2,g})$ . For more details see (3.20) and (3.23).

*Remark 1.6.* From Theorem 1.4, we can see that at  $x \in M_{sing}$ , we have  $|e^{-\frac{u}{p} \square_p}(x, x) - |G_x| \mathcal{L}im_u(x)| \leq Cp^{-1/2}$ . In particular, unlike in the usual non-singular case, if  $M_{sing}$  is not empty, the asymptotics of Theorem 1.3 cannot be uniform on  $M_{reg}$ .

This paper is organized as follows. In Section 2 we recall the definitions and basic properties of orbifolds. In Section 3 we study the convergence of the heat kernel and prove Theorems 1.3 and 1.4. Finally, in Section 4 we use these asymptotic result to prove the holomorphic Morse inequalities (Theorem 1.1).

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## 2 Background on orbifolds

In this section we recall the background about orbifold. The content of this section is essentially taken from [15] and [16].

### 2.1 Definitions

We first define a category  $\mathcal{M}_s$  as follows.

**Objects:** classes of pairs  $(G, M)$  where  $M$  is a connected smooth manifold and  $G$  is a finite group acting effectively on  $M$  (i.e., the unit is the unique element of  $G$  acting as  $\text{Id}_M$ );

**Morphisms:** a morphism  $\Phi: (G, M) \rightarrow (G', M')$  is a family of open embeddings  $\{\varphi: M \rightarrow M'\}_{\varphi \in \Phi}$  satisfying:

1. for each  $\varphi \in \Phi$  there is an injective group morphism  $\lambda_\varphi: G \hookrightarrow G'$  for which  $\varphi$  is equivariant, i.e.,  $\varphi(g.x) = \lambda_\varphi(g).\varphi(x)$  for  $x \in M$  and  $g \in G$ ;
2. for  $g' \in G'$  and  $\varphi \in \Phi$ , we have  $g'.(\varphi(M)) \cap \varphi(M) \neq \emptyset \implies g' \in \lambda_\varphi(G)$ ;
3. for any  $\varphi \in \Phi$ , we have  $\Phi = \{g'\varphi, g' \in G'\}$ , where  $g'\varphi: x \in M \mapsto g'.\varphi(x) \in M'$ .

**Definition 2.1** (Orbifold chart, atlas, structure). Let  $M$  be a paracompact Hausdorff space.

An  $m$ -dimensional *orbifold chart* on  $M$  consists of a connected open set  $U$  of  $M$ , an object  $(G_U, \tilde{U})$  of  $\mathcal{M}_s$  with  $\dim \tilde{U} = m$ , and a ramified covering  $\tau_U: \tilde{U} \rightarrow U$  which is  $G_U$ -invariant and induces a homeomorphism  $U \simeq \tilde{U}/G_U$ . We denote the chart by  $(G_U, \tilde{U}) \xrightarrow{\tau_U} U$ .

An  $m$ -dimensional *orbifold atlas*  $\mathcal{V}$  on  $M$  consists of a family of  $m$ -dimensional orbifold charts  $\mathcal{V}(U) = ((G_U, \tilde{U}) \xrightarrow{\tau_U} U)$  satisfying the following conditions :

1. the open sets  $U \subset M$  form a covering  $\mathcal{U}$  such that:

$$\text{for any } U, U' \in \mathcal{U} \text{ and } x \in U \cap U', \text{ there exists } U'' \in \mathcal{U} \text{ such that } x \in U'' \subset U \cap U'. \quad (2.1)$$

2. for any  $U, V \in \mathcal{U}$  with  $U \subset V$  there exists a morphism (of  $\mathcal{M}_s$ )  $\varphi_{VU}: (G_U, \tilde{U}) \rightarrow (G_V, \tilde{V})$  which covers the inclusion  $U \subset V$  and satisfies  $\varphi_{WU} = \varphi_{WV} \circ \varphi_{VU}$  for any  $U, V, W \in \mathcal{U}$  with  $U \subset V \subset W$ .

It is easy to see that there exists a unique maximal orbifold atlas  $\mathcal{V}_{max}$  containing  $\mathcal{V}$ : it consists of all orbifold charts  $(G_U, \tilde{U}) \xrightarrow{\tau_U} U$  which are locally isomorphic to charts from  $\mathcal{V}$  in the neighborhood of each point of  $M$ . A maximal orbifold atlas  $\mathcal{V}_{max}$  is called an *orbifold structure* and the pair  $(M, \mathcal{V}_{max})$  is called an orbifold. As usual, once we have an orbifold atlas  $\mathcal{V}$  on  $M$  we denote the orbifold by  $(M, \mathcal{V})$ , since  $\mathcal{V}$  uniquely determines  $\mathcal{V}_{max}$ .

In the above definition, we can replace  $\mathcal{M}_s$  with a category of manifolds with an additional structure such as orientation, Riemannian metric, almost-complex structure or complex structure. In this case we require that the morphisms and the groups preserve the specified structure. In this way we can define oriented, Riemannian, almost-complex or complex orbifolds.

Certainly, for any object  $(G_U, \tilde{U})$  of  $\mathcal{M}_s$ , we can always construct a  $G$ -invariant Riemannian metric on  $\tilde{U}$ . By a partition of unity argument, there always exists a Riemannian metric on an given orbifold  $(M, \mathcal{V})$ .

*Remark 2.2.* Let  $P$  be a smooth manifold, and let  $H$  be a compact Lie group acting locally freely on  $P$ . Then the quotient space  $P/H$  is an orbifold. Reciprocally, any orbifold  $M$  can be presented by this way, with  $H = O(n)$  ( $n = \dim M$ ) see [12, p. 76] and [13, p. 144].

**Definition 2.3** (Regular and singular set). Let  $(M, \mathcal{V})$  be an orbifold. For each  $x \in M$ , we can choose a small neighborhood  $(G_x, \tilde{U}_x) \rightarrow U_x$  such that  $x \in \tilde{U}_x$  is a fixed point of  $G_x$  (such  $G_x$  is unique up to isomorphisms for each  $x \in M$  from the definition). If the cardinal  $|G_x|$  of  $G_x$  is 1, then  $x$  is a regular point of  $M$ , meaning that  $M$  is a smooth manifold in a neighborhood of  $x$ . If  $|G_x| > 1$ , then  $x$  is a singular point of  $M$ . We denote by  $M_{sing} = \{x \in M : |G_x| > 1\}$  the singular set of  $M$  and by  $M_{reg} = M \setminus M_{sing}$  the regular set of  $M$ .

The following lemma is proved in [15, Lem. 5.4.3].

**Lemma 2.4.** *With the above notations, we can choose the local coordinates  $\tilde{U}_x \subset \mathbb{R}^m$  such that the finite group  $G_x$  acts linearly on  $\mathbb{R}^m$  and  $\{0\} = \tau_x^{-1}(x)$ .*

*In the sequel we will always use such charts.*

**Definition 2.5** (Orbifold vector bundle). An orbifold vector bundle  $E$  over an orbifold  $(M, \mathcal{V})$  is defined as follows:  $E$  is an orbifold and for  $U \in \mathcal{U}$ ,  $(G_U^E, \tilde{p}_U: \tilde{E}_U \rightarrow \tilde{U})$  is a  $G_U^E$ -equivariant vector bundle where  $(G_U^E, \tilde{E}_U)$  gives the orbifold structure of  $E$  and  $(G_U = G_U^E/K_U^E, \tilde{U})$ ,  $K_U^E = \ker(G_U^E \rightarrow \text{Diffeo}(\tilde{U}))$ , gives the orbifold structure on  $M$ .

If  $G_U^E$  acts effectively on  $\tilde{U}$  for  $U \in \mathcal{U}$ , i.e.  $K_U^E = \{1\}$ , we call  $E$  a proper orbifold vector bundle.

Let  $E$  be an orbifold vector bundle on  $(M, \mathcal{V})$ . For  $U \in \mathcal{U}$ , let  $\widetilde{E}_U^{pr}$  be the maximal  $K_U^E$ -invariant sub-bundle of  $\widetilde{E}_U$  on  $U$ . Then  $(G_U, \widetilde{E}_U^{pr})$  defines a proper orbifold vector bundle on  $(M, \mathcal{V})$ , which is denoted by  $E^{pr}$ .

*Example 2.6.* The (proper) orbifold tangent bundle  $TM$  of an orbifold  $M$  is defined by  $(G_U, T\widetilde{U} \rightarrow \widetilde{U})$ , for  $U \in \mathcal{U}$ .

**Definition 2.7** ( $\mathcal{C}^k$  section). Let  $E \rightarrow M$  be an orbifold bundle. A section  $s: M \rightarrow E$  is called  $\mathcal{C}^k$ , for  $k \in \mathbb{N} \cup \{\infty\}$ , if for each  $U \in \mathcal{U}$ ,  $s|_U$  is covered by a  $G_U^E$ -invariant smooth (or  $\mathcal{C}^k$ ) section  $\widetilde{s}_U: \widetilde{U} \rightarrow \widetilde{E}_U$ . We denote by  $\mathcal{C}^k(M, E)$  the space of  $\mathcal{C}^k$  sections of  $E$  on  $M$ .

**When it entails no confusion, we will denote  $\widetilde{s}_U$  simply by  $s$ .**

*Remark 2.8.* A smooth object on  $M$  as a section, a Riemannian metric, a complex structure, etc... can be seen as an usual regular object on  $M_{reg}$  such that its lift in any chart of the orbifold atlas can be extended to a smooth corresponding object.

**Definition 2.9** (Integration). If  $M$  is oriented, we define the integral  $\int_M \omega$  for a form  $\omega$  over  $M$  (i.e. a section of  $\Lambda(T^*M)$  over  $M$ ) as follows: if  $\text{supp}(\omega) \subset U \in \mathcal{U}$ , then

$$\int_M \omega = \frac{1}{|G_U|} \int_{\widetilde{U}} \widetilde{\omega}_U. \quad (2.2)$$

It is easy to see that the definition is independent of the chart. For general  $\omega$  we extend the definition by using a partition of unity.

Note also that if  $M$  is a Riemannian orbifold, there exists a canonical volume element  $dv_M$  on  $M$ , which is a section of  $\Lambda^{\dim M}(T^*M)$ . Hence, we can also integrate functions on  $M$ .

**Definition 2.10** (Metric structure on Riemannian orbifold). Let  $(M, \mathcal{V})$  be a compact Riemannian orbifold. For  $x, y \in M$ , we define  $d^M(x, y)$  by:

$$\begin{aligned} d^M(x, y) = \inf_{\gamma} \left\{ \sum_i \int_{t_{i-1}}^{t_i} \left| \frac{\partial}{\partial t} \widetilde{\gamma}_i(t) \right| dt \mid \gamma: [0, 1] \rightarrow M, \gamma(0) = x, \gamma(1) = y, \text{ such that} \right. \\ \left. \text{there are } t_0 = 0 < t_1 < \dots < t_k = 1 \text{ with } \gamma([t_{i-1}, t_i]) \subset U_i, U_i \in \mathcal{U}, \right. \\ \left. \text{and a } \mathcal{C}^\infty \text{ map } \widetilde{\gamma}_i: [t_{i-1}, t_i] \rightarrow \widetilde{U}_i \text{ which covers } \gamma|_{[t_{i-1}, t_i]} \right\}. \end{aligned} \quad (2.3)$$

Then  $(M, d)$  is a metric space.

## 2.2 Kernels on orbifolds

Let  $(M, \mathcal{V})$  be a Riemannian orbifold and let  $E$  be a proper orbifold vector bundle on  $M$ .

For any orbifold chart  $(G_U, \widetilde{U}) \xrightarrow{\tau_U} U$ ,  $U \in \mathcal{U}$ , we will add a tilda  $\sim$  to objects on  $U$  to indicate the corresponding objects on  $\widetilde{U}$ .

Consider a section  $\widetilde{\mathcal{K}} \in \mathcal{C}^\infty(\widetilde{U} \times \widetilde{U}, \text{pr}_1^* \widetilde{E} \otimes \text{pr}_2^* \widetilde{E}^*)$  such that

$$(g, 1) \widetilde{\mathcal{K}}(g^{-1} \widetilde{x}, \widetilde{x}') = (1, g^{-1}) \widetilde{\mathcal{K}}(\widetilde{x}, g \widetilde{x}') \quad \text{for any } g \in G_U, \quad (2.4)$$

where the action of  $G_U \times G_U$  on  $\widetilde{E}_{\widetilde{x}} \otimes \widetilde{E}_{\widetilde{x}'}^*$  is given by  $(g_1, g_2).u \otimes \xi = (g_1 u) \otimes (g_2 \xi)$ . We can then define an operator  $\widetilde{\mathcal{K}}: \mathcal{C}_0^\infty(\widetilde{U}, \widetilde{E}) \rightarrow \mathcal{C}^\infty(\widetilde{U}, \widetilde{E})$  by

$$(\widetilde{\mathcal{K}} \widetilde{s})(\widetilde{x}) = \int_{\widetilde{U}} \widetilde{\mathcal{K}}(\widetilde{x}, \widetilde{x}') \widetilde{s}(\widetilde{x}') dv_{\widetilde{U}}(\widetilde{x}') \quad \text{for } \widetilde{s} \in \mathcal{C}_0^\infty(\widetilde{U}, \widetilde{E}). \quad (2.5)$$

Recall that a section  $s \in \mathcal{C}^\infty(U, E)$  is identified with a  $G_U$ -invariant section  $\tilde{s} \in \mathcal{C}^\infty(\tilde{U}, \tilde{E})$ . Thus, we can define an operator  $\mathcal{K}: \mathcal{C}_0^\infty(U, E) \rightarrow \mathcal{C}^\infty(U, E)$  by

$$(\mathcal{K}s)(x) = \frac{1}{|G_U|} \int_{\tilde{U}} \tilde{\mathcal{K}}(\tilde{x}, \tilde{x}') \tilde{s}(\tilde{x}') dv_{\tilde{U}}(\tilde{x}') \quad \text{for } s \in \mathcal{C}_0^\infty(U, E), \quad (2.6)$$

where  $\tilde{x} \in \tau_U^{-1}(x)$ . Then the smooth kernel  $\mathcal{K}(x, x')$  of the operator  $\mathcal{K}$  with respect to  $dv_M$  is given by

$$\mathcal{K}(x, x') = \sum_{g \in G_U} (g, 1) \tilde{\mathcal{K}}(g^{-1}\tilde{x}, \tilde{x}'). \quad (2.7)$$

### 2.3 Complex orbifolds and Dolbeault cohomology

Let  $M$  be a compact complex orbifold of complex dimension  $n$  and with complex structure  $J$ . Let  $E$  be a holomorphic orbifold vector bundle on  $M$ .

Let  $\mathcal{O}_M$  be the sheaf over  $M$  of local  $G_U$ -invariant holomorphic functions over  $\tilde{U}$ , for  $U \in \mathcal{U}$ . Likewise, the local  $G_U^E$ -invariant sections of  $\tilde{E}$  over  $\tilde{U}$  define a sheaf  $\mathcal{O}_M(E)$  over  $M$ . Let  $H^\bullet(M, \mathcal{O}_M(E))$  be the cohomology of this sheaf. Notice that, by the definition, we have  $\mathcal{O}_M(E) = \mathcal{O}_M(E^{pr})$ . Thus without lost generality, we may and will assume that  $E$  is a proper orbifold vector bundle on  $M$ .

Consider a section  $s \in \mathcal{C}^\infty(M, E)$  and a local section  $\tilde{s} \in \mathcal{C}^\infty(\tilde{U}, \tilde{E})$  covering  $s$  over  $U$ . Then  $\bar{\partial} \tilde{s}$  covers a section of  $T^{*(0,1)}X \otimes E$  over  $U$ , denoted by  $\bar{\partial}^E s|_U$ . The sections  $\bar{\partial}^E s|_U$  for  $U \in \mathcal{U}$  patch together to define a global section  $\bar{\partial}^E s$  of  $T^{*(0,1)}X \otimes E$  over  $M$ . In a similar way, we can define  $\bar{\partial}^E \alpha$  for  $\alpha \in \Omega^{\bullet, \bullet}(M, E) := \mathcal{C}^\infty(M, \Lambda^{\bullet, \bullet}(T^*M) \otimes E)$ . We thus obtain the Dolbeault complex

$$0 \rightarrow \Omega^{0,0}(M, E) \xrightarrow{\bar{\partial}^E} \dots \xrightarrow{\bar{\partial}^E} \Omega^{0,n}(M, E) \rightarrow 0. \quad (2.8)$$

From the abstract de Rham theorem, there exists a canonical isomorphism (for more details, see [15, Sect. 5.4.2])

$$H^\bullet(\Omega^{0, \bullet}(M, E), \bar{\partial}^E) \simeq H^\bullet(M, \mathcal{O}_M(E)). \quad (2.9)$$

In the sequel, we will denote both these cohomology group simply by  $H^\bullet(M, E)$ .

Let  $g^{TM}$  be a Riemannian metric on  $TM$ , with associated volume form  $dv_M$ , and let  $h^E$  be a Hermitian metric on  $E$ . They induce a  $L^2$  Hermitian product  $\langle \cdot, \cdot \rangle$  on  $\Omega^\bullet(M, E)$  given by

$$\langle s_1, s_2 \rangle = \int_M \langle s_1(x), s_2(x) \rangle_{\Lambda^{0, \bullet}(T^*X) \otimes E} dv_M(x). \quad (2.10)$$

Let  $\bar{\partial}^{E,*}$  be the formal adjoint of  $\bar{\partial}^E$  for this  $L^2$  product, and let

$$\begin{aligned} D^E &= \sqrt{2}(\bar{\partial}^E + \bar{\partial}^{E,*}), \\ \square^E &= \frac{1}{2}D^{E,2} = \bar{\partial}^E \bar{\partial}^{E,*} + \bar{\partial}^{E,*} \bar{\partial}^E. \end{aligned} \quad (2.11)$$

be, respectively, the associated Dolbeault-Dirac operator and Kodaira Laplacian. Then  $\square^E$  a differential operator of order 2 acting on sections of  $\Lambda(T^*M) \otimes E$  (i.e., on each  $U$ , it is covered by a  $G_U^{\Lambda(T^*M) \otimes E}$ -invariant differential operator of order 2 acting on  $\mathcal{C}^\infty(\tilde{U}, \Lambda(\tilde{T}^*\tilde{M}) \otimes \tilde{E})$ ), which is formally self-adjoint and elliptic. Moreover, it preserves the  $\mathbb{Z}$ -grading on  $\Omega^\bullet(M, E)$ .

A crucial point is that classical Hodge theory still holds in the present orbifold setting, see [14, Prop. 2.2].



**Theorem 2.11** (Hodge theory). *For any  $q \in \mathbb{N}$ , we have the following orthogonal decomposition*

$$\Omega^{0,q}(M, E) = \text{Ker}(D^E|_{\Omega^{0,q}}) \oplus \text{Im}(\bar{\partial}^E|_{\Omega^{0,q-1}}) \oplus \text{Im}(\bar{\partial}^{E,*}|_{\Omega^{0,q+1}}). \quad (2.12)$$

*In particular, for  $q \in \mathbb{N}$ , we have the canonical isomorphism*

$$\text{Ker}(D^E|_{\Omega^{0,q}}) = \text{Ker}(\square^E|_{\Omega^{0,q}}) \simeq H^q(M, E). \quad (2.13)$$

Note that on orbifolds, we still have a unique Hermitian and holomorphic connection  $\nabla^E$  associated with  $E$  and  $h^E$ . We call it the Chern connection of  $(E, h^E)$ .

### 3 Convergence of the heat kernel

In the sequel, when we define objects on orbifolds, one can always think of them as being defined by standard objects which are  $G_U$ -invariant on each chart  $(G_U, \tilde{U})$ .

Let  $M$  be a compact complex orbifold of dimension  $n$ . We denote the complex structure of  $M$  by  $J$ . Let  $(L, h^L)$  an orbifold Hermitian holomorphic line bundle on  $M$  and let  $(E, h^E)$  be an orbifold Hermitian holomorphic vector bundle on  $M$ . Recall that by Section 2.3 we may assume that  $L$  and  $E$  are proper. We denote the corresponding Chern connections by  $\nabla^L$  and  $\nabla^E$  respectively, and we denote their curvatures by  $R^L$  and  $R^E$ .

We define the orbifold bundles  $\mathbb{E}$  and  $\mathbb{E}_p$  over  $M$  by

$$\begin{aligned} \mathbb{E} &= \Lambda^{0,\bullet}(T^*M) \otimes E, \\ \mathbb{E}_p &= \Lambda^{0,\bullet}(T^*M) \otimes E \otimes L^p. \end{aligned} \quad (3.1)$$

Let  $g^{TM}$  be a Riemannian metric on  $TM$  which is compatible with  $J$ . Then  $\mathbb{E}$  and  $\mathbb{E}_p$  are naturally equipped with the Hermitian metrics  $h^{\mathbb{E}}$  and  $h^{\mathbb{E}_p}$  induced by  $g^{TM}$ ,  $h^L$  and  $h^E$ . We endow  $\mathcal{C}^\infty(M, \mathbb{E}_p)$  with the  $L^2$  scalar product associated with  $g^{TM}$ ,  $h^L$  and  $h^E$  as in (2.10). As in section 2.3, we can define  $D^{L^p \otimes E}$  and  $\square^{L^p \otimes E}$ . We will denote these operators respectively by  $D_p$  and  $\square_p$  for short.

Let  $e^{-u\square_p}$  be the heat kernel of  $\square_p$  and let  $e^{-u\square_p}$  be its smooth kernel with respect to  $dv_M(x')$ . Concerning heat kernels on orbifolds, we refer the reader to [14, Sect. 2.1].

In this section, we study the convergence as  $p \rightarrow +\infty$  of the heat kernel. We follow the approach of [15].

**Localisation** Let  $\varepsilon > 0$  be a small number (smaller than a quarter of the injectivity radius of  $(M, g^{TM})$ ). Let  $f: \mathbb{R} \rightarrow [0, 1]$  be a smooth even function such that

$$f(t) = \begin{cases} 1 & \text{for } |t| < \varepsilon/2, \\ 0 & \text{for } |t| > \varepsilon. \end{cases} \quad (3.2)$$

For  $u > 0$  and  $a \in \mathbb{C}$ , set

$$\begin{aligned} F_u(a) &= \int_{\mathbb{R}} e^{iva} \exp(-v^2/2) f(v\sqrt{u}) \frac{dv}{\sqrt{2\pi}}, \\ G_u(a) &= \int_{\mathbb{R}} e^{iva} \exp(-v^2/2) (1 - f(v\sqrt{u})) \frac{dv}{\sqrt{2\pi}}. \end{aligned} \quad (3.3)$$

These functions are even holomorphic functions. Moreover, the restrictions of  $F_u$  and  $G_u$  to  $\mathbb{R}$  lie in the Schwartz space  $\mathcal{S}(\mathbb{R})$ , and

$$F_u(vD_p) + G_u(vD_p) = \exp\left(-\frac{v^2}{2} D_p^2\right) \text{ for } v > 0. \quad (3.4)$$

Let  $G_u(vD_p)(x, x')$  be the smooth kernel of  $G_u(vL_p)$  with respect to  $dv_M(x')$ . Then for any  $m \in \mathbb{N}$ ,  $u_0 > 0$ ,  $\varepsilon > 0$ , there exist  $C > 0$  and  $N \in \mathbb{N}$  such that for any  $u > u_0$  and any  $p \in \mathbb{N}^*$ ,

$$\left| G_{\frac{u}{p}} \left( \sqrt{u/p} D_p \right) (\cdot, \cdot) \right|_{\mathcal{C}^m(M \times M)} \leq Cp^N \exp \left( -\frac{\varepsilon^2 p}{16u} \right). \quad (3.5)$$

This is proved in the same way as [15, Prop. 1.6.4], the only difference is that when we use Sobolev norms or inequality, we have to use it on  $\tilde{U}$  for the pulled-back operators.

As pointed out in [14], the property of the finite propagation speed of solutions of hyperbolic equations still holds on an orbifold (see the proof in [15, App. D.2]). Thus,  $F_{\frac{u}{p}} \left( \sqrt{u/p} D_p \right) (x, x')$  vanishes if  $d^M(x, x') \geq \varepsilon$  and  $F_{\frac{u}{p}} \left( \sqrt{u/p} D_p \right) (x, \cdot)$  only depends on the restriction of  $D_p$  to the ball  $B^M(x, \varepsilon)$ . This, together with (3.4) and (3.5), implies that the problem of the asymptotic of  $e^{-u\Box_p}(x, \cdot)$  is local.

More precisely, this means that, for  $x_0 \in M$  fixed, one can trivialize the various bundles over  $U_{x_0}$  and replace  $M$  by  $M_0 = \mathbb{C}^n / G_{x_0} \supset \tilde{U}_{x_0} / G_{x_0} = U_{x_0}$  (using a local chart as in Lemma 2.4). Then we construct a metric  $g^{TM_0}$  on  $M_0$  and an operator  $L_{p, x_0}$  acting on  $\mathbb{E}_{p, x_0}$  over  $M_0$  such that  $g^{TM_0}$  (resp.  $L_{p, x_0}$ ) coincides with  $g^{TM}$  (resp.  $\Box_p$ ) near  $x_0 = 0$  and such that its lift  $\widetilde{g^{TM_0}}$  (resp.  $\tilde{L}_{p, x_0}$ ) on  $\tilde{M}_0 = \mathbb{C}^n$  is the constant metric  $(\widetilde{g^{TM_0}})_{\tilde{x}=0}$  (resp. the usual Laplacian on  $\mathbb{C}^n$  for this metric) away from 0. Then we can approximate the heat kernel of  $\Box_p$  by the one of  $L_{p, x_0}$ , see equation (3.13) below.

We now give the details of these constructions, following [15, Sect 1.6.3].

By [15, (1.2.61) and (1.4.27)], the Levi-Civita connection  $\nabla^{TM}$  on  $(M, g^{TM})$ , the complex structure  $J$  of  $M$  and the Chern connection  $\nabla^E$  induce a Hermitian connexion  $\nabla^{B, \mathbb{E}}$  on  $(\mathbb{E}, h^{\mathbb{E}})$  which preserve the  $\mathbb{Z}$ -grading. This connection is called the Bismut connection. Let  $\Delta^{B, \mathbb{E}}$  be the associated Bochner Laplacian, that is

$$\Delta^{B, \mathbb{E}} = - \sum_{i=1}^{2n} \left( (\nabla_{e_i}^{B, \mathbb{E}})^2 - \nabla_{\nabla_{e_i}^{TM} e_i}^{B, \mathbb{E}} \right), \quad (3.6)$$

with  $\{e_i\}$  an orthonormal frame of  $TM$ .

Let  $\{w_j\}_{j=1}^n$  be a local orthonormal frame of  $T^{(1,0)}M$  (with the metric induced by  $g^{TM}$ ) with dual frame  $\{w^j\}_{j=1}^n$ . Set

$$\begin{aligned} \omega_d &= - \sum_{k, \ell} R^L(w_k, \bar{w}_\ell) \bar{w}^\ell \wedge i \bar{w}_k, \\ \tau &= \sum_j R^L(w_j, \bar{w}_j). \end{aligned} \quad (3.7)$$

From Bismut's Lichnerowicz formula (see [15, Thm. 1.4.7]), there exist a self-adjoint section  $\Phi_E$  of  $\text{End}(\Lambda^{0, \bullet}(T^*M) \otimes E)$  such that

$$\Box_p = \frac{1}{2} \Delta^{B, \mathbb{E}} + \omega_d + \frac{1}{2} \tau + \Phi_E. \quad (3.8)$$

We fix  $x_0 \in M$  and  $\varepsilon > 0$  smaller than the quarter of the injectivity radius of  $M$ . In the sequel we will use a local chart  $(G_{x_0}, \tilde{U}_{x_0})$  near  $x_0$  as in Lemma 2.4. Note that we then have  $T_{x_0}M \simeq \mathbb{C}^n / G_{x_0}$ .

We denote by  $B^M(x_0, 4\varepsilon)$  and  $B^{T_{x_0}M}(0, 4\varepsilon)$  the open balls in  $M$  and  $T_{x_0}M$  with center  $x_0$  and 0 and radius  $4\varepsilon$ , respectively. The exponential map  $T_{x_0}M \ni Z \mapsto \exp_{x_0}^M(Z) \in M$  is a

diffeomorphism from  $B^{T_{x_0}M}(0, 4\varepsilon)$  on  $B^M(x_0, 4\varepsilon)$ . From now on, we identify  $B^{T_{x_0}M}(0, \varepsilon)$  on  $B^M(x_0, 4\varepsilon)$ .

For  $Z \in B^{T_{x_0}M}(0, 4\varepsilon)$ , we identify  $(L_Z, h_Z^L)$  and  $(\mathbb{E}_Z, h_Z^\mathbb{E})$  to  $(L_{x_0}, h_{x_0}^L)$  and  $(\mathbb{E}_{x_0}, h_{x_0}^\mathbb{E})$  by the parallel transport with respect to  $\nabla^L$  and  $\nabla^{B, \mathbb{E}}$  along the ray  $u \in [0, 1] \mapsto uZ$ . We denote the corresponding connection forms by  $\Gamma^L$  and  $\Gamma^\mathbb{E}$ . Note that  $\Gamma^L$  and  $\Gamma^\mathbb{E}$  are skew-adjoint with respect to  $h_{x_0}^L$  and  $h_{x_0}^\mathbb{E}$ .

Let  $\rho: \mathbb{R} \rightarrow [0, 1]$  be a smooth even function such that

$$\rho(v) = 1 \text{ if } |v| < 2 \quad \text{and} \quad \rho(v) = 0 \text{ if } |v| > 4. \quad (3.9)$$

We denote by  $\nabla_{\tilde{V}}$  the ordinary differentiation operator on  $\tilde{M}_0 = \mathbb{C}^n$  in the direction  $\tilde{V}$ .  $\nabla$  is  $G_{x_0}$ -equivariant since  $G_{x_0}$  acts linearly on  $\mathbb{C}^n$ , thus it induces an operator still denoted by  $\nabla$  on  $M_0 = \mathbb{C}^n/G_{x_0}$ . Set

$$\nabla^{\mathbb{E}_{p, x_0}} = \nabla + \rho(|Z|/\varepsilon)(p\Gamma^L + \Gamma^\mathbb{E})(Z), \quad (3.10)$$

which is a Hermitian connection on the trivial bundle  $(\mathbb{E}_{p, x_0}, h_{x_0}^{\mathbb{E}_p})$  over  $M_0 \simeq T_{x_0}M$ , where the identification is given by

$$[Z_1, \dots, Z_{2n}] \in \mathbb{R}^{2n}/G_{x_0} \mapsto \left[ \sum_{i=1}^{2n} Z_i \tilde{e}_i \right] \in T_{x_0}M. \quad (3.11)$$

Here,  $\{\tilde{e}_i\}_i$  is an orthonormal basis of  $(\tilde{T}\tilde{M}_{U_{x_0}})_{\tilde{x}=0}$ .

Let  $g^{TM_0}$  be the metric on  $M_0$  which coincides with  $g^{TM}$  on  $B^{T_{x_0}M}(0, 2\varepsilon)$  and such that its lift  $\tilde{g}^{TM_0}$  on  $\tilde{M}_0$  is the constant metric  $(\tilde{g}^{TM}_{U_{x_0}})_{\tilde{x}=0}$  outside of  $B^{\tilde{M}_0}(0, 4\varepsilon)$ . Let  $dv_{\tilde{M}_0}$  be the Riemannian volume of  $\tilde{g}^{TM_0}$ .

Let  $\Delta^{\mathbb{E}_{p, x_0}}$  be the Bochner Laplacian associated with  $\nabla^{\mathbb{E}_{p, x_0}}$  and  $g^{TM_0}$ . Set

$$L_{p, x_0} = \frac{1}{2} \Delta^{\mathbb{E}_{p, x_0}} - p\rho(|Z|/\varepsilon)(\omega_{d, Z} + \frac{1}{2}\tau_Z) - \rho(|Z|/\varepsilon)\Phi_{E, Z}. \quad (3.12)$$

Then  $L_{p, x_0}$  is a self-adjoint operator for the  $L^2$ -product induced by  $h_{x_0}^{\mathbb{E}_p}$  and  $g^{TM_0}$ , and coincides with  $\square_p$  on  $B^{T_{x_0}M}(0, 2\varepsilon)$ . We denote by  $\tilde{L}_{p, x_0}$  the  $G_{x_0}$ -invariant lift of  $L_{p, x_0}$  on  $\tilde{M}_0$ .

From the above discussion, and using the same technics as in [15, Lem. 1.6.5] we can then prove that for  $(x, x') \in B^M(x_0, \varepsilon/2)$  corresponding to  $(Z, Z') \in M_0$ , there exist  $C > 0$  and  $k \in \mathbb{N}$  such that

$$\left| e^{-\frac{u}{p}\square_p}(x, x') - e^{-\frac{u}{p}L_{p, x_0}}(Z, Z') \right| \leq Cp^k e^{-\frac{\varepsilon^2 p}{16u}}, \quad (3.13)$$

where  $e^{-\frac{u}{p}L_{p, x_0}}(Z, Z')$  is the smooth kernel of  $e^{-\frac{u}{p}L_{p, x_0}}$  with respect to  $dv_{M_0}(Z')$ , the Riemannian volume of  $\tilde{g}^{TM_0}$ .

Now, as in Section 2.2, we have

$$e^{-\frac{u}{p}L_{p, x_0}}(Z, Z') = \sum_{g \in G_{x_0}} (g, 1) e^{-\frac{u}{p}\tilde{L}_{p, x_0}}(g^{-1}\tilde{Z}, \tilde{Z}'), \quad (3.14)$$

where  $e^{-\frac{u}{p}\tilde{L}_{p, x_0}}(\tilde{Z}, \tilde{Z}')$  is the smooth kernel of  $e^{-\frac{u}{p}\tilde{L}_{p, x_0}}$  with respect to  $dv_{\tilde{M}_0}(\tilde{Z}')$ .

Thus, from (3.13) and (3.14), we have to study the asymptotic of  $e^{-\frac{u}{p}\tilde{L}_{p, x_0}}(\tilde{Z}, \tilde{Z}')$ , which is a kernel on a honest vector space. Note that, even if we want to restrict  $e^{-\frac{u}{p}\square_p}$  to the diagonal, we have to study the off-diagonal asymptotic of  $e^{-\frac{u}{p}\tilde{L}_{p, x_0}}$ .

**Rescaling** As in [15], we will rescale the variables in  $\widetilde{M}_0$ .

In this paragraph, we work on  $\widetilde{U}_{x_0}$  and for any bundle  $F$  on  $M$  we will denote  $\widetilde{F}_{U_{x_0}}$  simply by  $\widetilde{F}$ . Likewise, we will drop the subscript  $U_{x_0}$  in the notations of lifts to  $\widetilde{F}$  of objects on  $F$ .

Let  $S_{\widetilde{L}}$  be a  $G_{x_0}$ -invariant unit vector of  $\widetilde{L}|_0$ . Using  $S_{\widetilde{L}}$  and the above discussion, we get an isometry  $\widetilde{\mathbb{E}}_{p,x_0} \simeq \widetilde{\mathbb{E}}_{x_0}$ . Thus,  $\widetilde{L}_{p,x_0}$  can be seen as an operator on  $\widetilde{\mathbb{E}}_{x_0}$ . Note that our formulas will not depend on the choice of  $S_L$  as the isomorphism  $\text{End}(\widetilde{\mathbb{E}}_{p,x_0}) \simeq \text{End}(\widetilde{\mathbb{E}}_{x_0})$  is canonical.

Let  $dv_{\widetilde{T}M}$  be the Riemannian volume of  $(\widetilde{M}_0, g^{\widetilde{T}M}|_0)$ . Let  $\widetilde{\kappa}$  be the smooth positive function defined by

$$dv_{\widetilde{M}_0}(\widetilde{Z}) = \widetilde{\kappa}(\widetilde{Z}) dv_{\widetilde{T}M}(\widetilde{Z}), \quad (3.15)$$

with  $\widetilde{\kappa}(0) = 1$ . Note that this definition is compatible with (1.13) near 0, which will be *in fine* the only region of interest.

Let  $R^{\widetilde{L}}$  be the Chern curvature of  $(\widetilde{L}, \widetilde{h}^{\widetilde{L}})$ . Let  $\widetilde{\omega}_d$  and  $\widetilde{\tau}$  be defined from  $R^{\widetilde{L}}$  as  $\omega_d$  and  $\tau$  were defined from  $R^L$  in (3.7). Then  $\widetilde{\omega}_d$  and  $\widetilde{\tau}$  are in fact the lifts of  $\omega_d$  and  $\tau$ .

Recall that  $\nabla_V$  is the ordinary differentiation operator on  $\widetilde{M}_0 = \mathbb{C}^n$  in the direction  $V$ .

We will now make the change of parameter  $t = \frac{1}{\sqrt{p}} \in ]0, 1]$ .

**Definition 3.1.** For  $s \in \mathcal{C}^\infty(\widetilde{M}_0, \mathbb{E}_{x_0})$  and  $\widetilde{Z} \in \mathbb{C}^n$  set

$$\begin{aligned} (S_t s)(\widetilde{Z}) &= s(\widetilde{Z}/t), \\ \nabla_0 &= \nabla + \frac{1}{2} R_0^{\widetilde{L}}(\widetilde{Z}, \cdot), \\ \widetilde{\mathcal{L}}_t &= t^2 S_t^{-1} \widetilde{\kappa}^{1/2} \widetilde{L}_{p,x_0} \widetilde{\kappa}^{-1/2} S_t, \\ \widetilde{\mathcal{L}}_0 &= -\frac{1}{2} \sum_{i=1}^{2n} (\nabla_{0,e_i})^2 - \widetilde{\omega}_{d,0} - \frac{1}{2} \widetilde{\tau}_0. \end{aligned} \quad (3.16)$$

Then, exactly as in [15, Lem. 1.6.6], our constructions imply that  $\widetilde{\mathcal{L}}_t = \widetilde{\mathcal{L}}_0 + O(t)$ .

Let  $e^{-u\widetilde{\mathcal{L}}_t}(\widetilde{Z}, \widetilde{Z}')$  be the smooth kernel of  $e^{-u\widetilde{\mathcal{L}}_t}$  with respect to  $dv_{\widetilde{T}M}(\widetilde{Z}')$  (for  $t > 0$  or  $t = 0$ ). Now, as we are working on a vector space, we can apply all the results of [15, Sects. 4.2.2] to  $\widetilde{\mathcal{L}}_t$  and  $\widetilde{\mathcal{L}}_0$ , and we get the following full off-diagonal convergence (see [15, Thm. 4.2.8]).

**Theorem 3.2.** *There exist  $C, C' > 0$  and  $N \in \mathbb{N}$  such that for any  $m, m' \in \mathbb{N}$  and  $u_0 > 0$ , there are  $C'' > 0$  and  $k \in \mathbb{N}$  such that for any  $t \in ]0, t_0]$ ,  $u \geq u_0$  and  $\widetilde{Z}, \widetilde{Z}' \in \widetilde{M}_0$  with  $|\widetilde{Z}|, |\widetilde{Z}'| \leq 1$*

$$\begin{aligned} \sup_{|\alpha|, |\alpha'| \leq m} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial \widetilde{Z}^\alpha \partial \widetilde{Z}'^{\alpha'}} \left( e^{-u\widetilde{\mathcal{L}}_t} - e^{-u\widetilde{\mathcal{L}}_0} \right) (\widetilde{Z}, \widetilde{Z}') \right|_{\mathcal{C}^{m'}(M)} \\ \leq C'' t (1 + |\widetilde{Z}| + |\widetilde{Z}'|)^N \exp \left( Cu - \frac{C'}{u} |\widetilde{Z} - \widetilde{Z}'|^2 \right), \end{aligned} \quad (3.17)$$

where  $|\cdot|_{\mathcal{C}^{m'}(M)}$  denotes the  $\mathcal{C}^{m'}$ -norm with respect to the parameter  $x_0 \in M$  used to define the operators  $\widetilde{\mathcal{L}}_t$  and  $\widetilde{\mathcal{L}}_0$  on  $\mathbb{C}^n$ .

**Conclusion** From (3.16), a change of variable gives that

$$e^{-\frac{u}{p} \widetilde{L}_{p,x_0}}(\widetilde{Z}, \widetilde{Z}') = p^n e^{-u\widetilde{\mathcal{L}}_t}(\widetilde{Z}/t, \widetilde{Z}'/t) \widetilde{\kappa}^{-1/2}(\widetilde{Z}) \widetilde{\kappa}^{-1/2}(\widetilde{Z}'). \quad (3.18)$$

Thus, with Theorem 3.2, we infer that for any multi-index  $\alpha$  with  $|\alpha| \leq m$  and for  $|\tilde{Z}|$  small,

$$\left| \frac{\partial^{|\alpha|}}{\partial \tilde{Z}^\alpha} \left( p^{-n} e^{-\frac{u}{p} \tilde{L}_{p,x_0}} (g^{-1} \tilde{Z}, \tilde{Z}) - e^{-u \tilde{\mathcal{L}}_0} (\sqrt{p} g^{-1} \tilde{Z}, \sqrt{p} \tilde{Z}) \tilde{\kappa}^{-1}(\tilde{Z}) \right) \right|_{\mathcal{C}^{m'}(M)} \leq C p^{\frac{m-1}{2}} (1 + \sqrt{p} |\tilde{Z}|)^N e^{-cp |\tilde{Z} - g^{-1} \tilde{Z}|^2}. \quad (3.19)$$

We define  $\dot{R}^{\tilde{L}} \in \text{End}(T^{(1,0)} \tilde{M}_0)$  by  $g^{TM}(\dot{R}^{\tilde{L}} u, \bar{v}) = \dot{R}^{\tilde{L}}(u, \bar{v})$  for  $u, v \in T^{(1,0)} \tilde{M}_0$ . We extend  $\dot{R}^{\tilde{L}}$  to  $T \tilde{M}_0 \otimes \mathbb{C} = T^{(1,0)} \tilde{M}_0 \oplus T^{(0,1)} \tilde{M}_0$  by setting  $\dot{R}^{\tilde{L}} \bar{v} = -\overline{\dot{R}^{\tilde{L}} v}$ . Then  $\sqrt{-1} \dot{R}_x^{\tilde{L}}$  induces an anti-symmetric endomorphism of  $T \tilde{M}_0$ . Observe that from the formula for the heat kernel of a harmonic oscillator (see [15, (E.2.4), (E.2.5)] for instance), we find:

$$e^{-u \tilde{\mathcal{L}}_0} (g^{-1} \tilde{Z}, \tilde{Z}) = \frac{1}{(2\pi)^n} \frac{\det(\dot{R}_0^{\tilde{L}}) e^{u \tilde{\omega}_{d,0}}}{\det(1 - \exp(-u \dot{R}_0^{\tilde{L}}))} \otimes \text{Id}_{\tilde{E}_0} \times \exp \left\{ - \left\langle \frac{\dot{R}_0^{\tilde{L}}/2}{\text{th}(u \dot{R}_0^{\tilde{L}}/2)} \tilde{Z}, \tilde{Z} \right\rangle + \left\langle \frac{\dot{R}_0^{\tilde{L}}/2}{\text{sh}(u \dot{R}_0^{\tilde{L}}/2)} e^{u \dot{R}_0^{\tilde{L}}/2} g^{-1} \tilde{Z}, \tilde{Z} \right\rangle \right\}. \quad (3.20)$$

Here, we use the convention that if an eigenvalue of  $\dot{R}_0^{\tilde{L}}$  is zero, then its contribution to the above term is  $\tilde{v} \mapsto \frac{1}{2\pi u} e^{-\frac{1}{2u} |g^{-1} \tilde{v} - \tilde{v}|^2}$ .

We are now able to prove Theorems 1.3 and 1.4.

*Proof of Theorem 1.3.* If  $x_0$  is in  $M_{reg}$ , we have  $G_{x_0} = \{1\}$  and the tilded and un-tilded objects coincide. Thus, from (3.13), (3.14), and (3.19), (3.20) applied at  $\tilde{Z} = 0$ , we get Theorem 1.3.  $\square$

*Proof of Theorem 1.4.* We will use here the notations given in the introduction of this paper (before the statement of Theorem 1.4). In Particular, for  $g \in G_{x_0}$ , we have a decomposition  $\tilde{Z} = (\tilde{Z}_{1,g}, \tilde{Z}_{2,g})$  where  $\tilde{Z}_{1,g}$  is in the fixed-point set of  $g$  and  $\tilde{Z}_{2,g}$  is in the normal bundle of this set.

Let us fix  $g \in G_{x_0}$ . The idea is to apply the results of this Section 3 but replacing the base-point  $0 \in \tilde{U}_{x_0}$  by  $\tilde{Z}_{1,g}$ . In order to stress on the dependence on  $\tilde{Z}_{1,g}$ , we will add subscript to the various objects introduced above but defined with the base-point  $\tilde{Z}_{1,g}$ , e.g.,  $\kappa_{\tilde{Z}_{1,g}}$ ,  $\tilde{\mathcal{L}}_{0,\tilde{Z}_{1,g}}$ , etc... We can then make (3.19) more precise: observe that there is  $c_0 > 0$  such that for each  $g \in G_{x_0}$ ,  $|\tilde{Z}_{2,g} - g^{-1} \tilde{Z}_{2,g}| \geq c_0 |\tilde{Z}_{2,g}|$ , and thus for  $m, \ell \in \mathbb{N}$  and  $|\alpha'| \leq \ell$  we have some constants  $c, C > 0$  such that

$$\sup_{|\alpha| \leq m} \left| \frac{\partial^{|\alpha|}}{\partial \tilde{Z}_{1,g}^\alpha} \frac{\partial^{|\alpha'|}}{\partial \tilde{Z}_{2,g}^{\alpha'}} \left( p^{-n} e^{-\frac{u}{p} \tilde{L}_{p,x_0}} (g^{-1} \tilde{Z}, \tilde{Z}) - e^{-u \tilde{\mathcal{L}}_{0,\tilde{Z}_{1,g}}} (\sqrt{p} g^{-1} \tilde{Z}_{2,g}, \sqrt{p} \tilde{Z}_{2,g}) \tilde{\kappa}_{\tilde{Z}_{1,g}}^{-1}(\tilde{Z}_{2,g}) \right) \right| \leq C p^{\frac{\ell-1}{2}} (1 + \sqrt{p} |\tilde{Z}_{2,g}|)^N e^{-cp |\tilde{Z}_{2,g}|^2}. \quad (3.21)$$

In particular, we find that if  $g = 1$  then  $\tilde{Z} = \tilde{Z}_{1,g}$  and  $\tilde{Z}_{2,g} = 0$  and thus

$$\sup_{|\alpha| \leq m} \left| \frac{\partial^{|\alpha|}}{\partial \tilde{Z}^\alpha} \left( p^{-n} e^{-\frac{u}{p} \tilde{L}_{p,x_0}} (\tilde{Z}, \tilde{Z}) - e^{-u \tilde{\mathcal{L}}_{0,\tilde{Z}}} (0, 0) \right) \right| \leq C p^{-1/2}. \quad (3.22)$$

Note that the image in the quotient of the union  $\cup_{g \neq 1} \tilde{U}_{x_0}^g$  is precisely  $M_{sing} \cap U_{x_0}$ . In particular, if  $Z$  is the image of  $\tilde{Z}$ , then we have  $|\tilde{Z}_{2,g}| \geq d(Z, M_{sing})$  for  $g \in G_{x_0} \setminus \{1\}$ . From this

remark and equations (3.13), (3.14), (3.21) and (3.22) we find that

$$\begin{aligned} \sup_{|\alpha| \leq \ell} \left| \frac{\partial^{|\alpha|}}{\partial \tilde{Z}^\alpha} \left( p^{-n} e^{-\frac{u}{p} \square_p} (\tilde{Z}, \tilde{Z}) - e^{-u \tilde{\mathcal{Z}}_{0,\tilde{Z}}} (0, 0) \right. \right. \\ \left. \left. - \sum_{\substack{g \in G_{x_0} \\ g \neq 1}} (g, 1) \cdot e^{-u \tilde{\mathcal{Z}}_{0,\tilde{Z}_{1,g}}} (\sqrt{p} g^{-1} \tilde{Z}_{2,g}, \sqrt{p} \tilde{Z}_{2,g}) \tilde{\kappa}_{\tilde{Z}_{1,g}}^{-1}(\tilde{Z}_{2,g}) \right) \right| \\ \leq C p^{-1/2} + C p^{\frac{\ell-1}{2}} (1 + \sqrt{p} d(Z, M_{sing}))^N e^{-c p d(Z, M_{sing})^2}. \end{aligned} \quad (3.23)$$

To conclude, we get Theorem 1.4 thanks to the definition of  $e^{i\theta_g}$  and  $g^E$ , and (1.11), (1.14), (3.20) and (3.23), noticing that  $\dot{R}^{\tilde{L}}$  and  $\tilde{\omega}_d$  coincide with the invariant lifts of  $\dot{R}^L$  and  $\omega_d$ .  $\square$

## 4 Proof of the inequalities

In this section, we will first prove Theorem 1.2, and then show how to use it in conjunction with the convergence of the heat kernel proved in Section 3 to get Theorem 1.1. The method is inspired by [2] (see also [15, Sect. 1.7]).

*Proof of Theorem 1.2.* If  $\lambda$  is an eigenvalue of  $\square_p$  acting on  $\Omega^{0,j}(M, L^p \otimes E)$ , we denote by  $F_j^\lambda$  the corresponding finite-dimensional eigenspace. As  $\bar{\partial}^{L^p \otimes E}$  and  $\bar{\partial}^{L^p \otimes E, *}$  commute with  $\square_p$ , we deduce that

$$\bar{\partial}^{L^p \otimes E}(F_j^\lambda) \subset F_{j+1}^\lambda \quad \text{and} \quad \bar{\partial}^{L^p \otimes E, *}(F_j^\lambda) \subset F_{j-1}^\lambda. \quad (4.1)$$

As a consequence, we have a complexe

$$0 \longrightarrow F_0^\lambda \xrightarrow{\bar{\partial}^{L^p \otimes E}} F_1^\lambda \xrightarrow{\bar{\partial}^{L^p \otimes E}} \cdots \xrightarrow{\bar{\partial}^{L^p \otimes E}} F_n^\lambda \longrightarrow 0. \quad (4.2)$$

If  $\lambda = 0$ , we have  $F_j^0 \simeq H^j(M, L^p \otimes E)$  by Theorem 2.11. If  $\lambda > 0$ , then the complex (4.2) is exact. Indeed, if  $\bar{\partial}^{L^p \otimes E} s = 0$  and  $s \in F_j^\lambda$ , then

$$s = \lambda^{-1} \square_p s = \lambda^{-1} \bar{\partial}^{L^p \otimes E} \bar{\partial}^{L^p \otimes E, *} s \in \text{Im}(\bar{\partial}^{L^p \otimes E}). \quad (4.3)$$

In particular, we get for  $\lambda > 0$  and  $0 \leq q \leq n$

$$\sum_{j=0}^q (-1)^{q-j} \dim F_j^\lambda = \dim (\bar{\partial}^{L^p \otimes E}(F_q^\lambda)) \geq 0, \quad (4.4)$$

with equality if  $q = n$ .

Now, by Theorem 2.11

$$\text{Tr}_{|\Omega^j|} [e^{-\frac{u}{p} \square_p}] = \dim(H^j(M, L^p \otimes E)) + \sum_{\lambda > 0} e^{-\frac{u}{p} \lambda} \dim F_j^\lambda. \quad (4.5)$$

Finally, (4.4) and (4.5) entail (1.9).  $\square$

We can now conclude.

*Proof of Theorem 1.1.* Let  $\{x_i\}_{1 \leq i \leq m}$  be a finite set of points of  $M_{sing}$  such that the corresponding local chart  $(G_{x_i}, \tilde{U}_{x_i})$  with  $\tilde{U}_{x_i} \subset \mathbb{C}^n$  satisfy

$$B^{\tilde{U}_{x_i}}(0, 2\varepsilon) \subset \tilde{U}_{x_i} \quad \text{and} \quad M_{sing} \subset \bigcup_{i=1}^m W_i, \quad W_i := B^{\tilde{U}_{x_i}}(0, \frac{\varepsilon}{4})/G_{x_i}. \quad (4.6)$$

Here  $\varepsilon$  is as in Section 3. Let  $W_0$  be an open neighborhood of the complementary of  $\bigcup_{i=1}^m W_i$  which is relatively compact in  $M_{reg}$ . Let  $\{\psi_k\}_{0 \leq k \leq m}$  be a partition of the unity subordinated to  $\{W_k\}_{0 \leq k \leq m}$ .

In the sequel, we denote by  $\text{Tr}_{\Lambda^{0,q}}$  the trace on either  $\Lambda^{0,q}(T^*M) \otimes L^p \otimes E$  or  $\Lambda^{0,q}(T^*M)$ . For  $0 \leq q \leq n$ , we have

$$\begin{aligned} \text{Tr}_q [e^{-\frac{u}{p}\square_p}] &= \int_M \text{Tr}_{\Lambda^{0,q}} [e^{-\frac{u}{p}\square_p}(x, x)] dv_M(x) \\ &= \sum_{k=0}^m \int_M \psi_k(x) \text{Tr}_{\Lambda^{0,q}} [e^{-\frac{u}{p}\square_p}(x, x)] dv_M(x). \end{aligned} \quad (4.7)$$

From Theorem 1.3, we know that for  $p \rightarrow \infty$

$$\begin{aligned} p^{-n} \int_M \psi_0(x) \text{Tr}_{\Lambda^{0,q}} [e^{-\frac{u}{p}\square_p}(x, x)] dv_M(x) &= \\ &= \frac{\text{rk}(E)}{(2\pi)^n} \int_M \psi_0(x) \frac{\det(\dot{R}_x^L) \text{Tr}_{\Lambda^{0,q}} [e^{u\omega_{d,x}}]}{\det(1 - \exp(-u\dot{R}_x^L))} dv_M(x) + o(1). \end{aligned} \quad (4.8)$$

For  $1 \leq k \leq m$ , we know from Theorem 1.4 that for  $p \rightarrow \infty$

$$\begin{aligned} p^{-n} \int_M \psi_k(x) \text{Tr}_{\Lambda^{0,q}} [e^{-\frac{u}{p}\square_p}(x, x)] dv_M(x) &= \\ &= \sum_{\substack{g \in G_x \\ g \neq 1}} \frac{1}{|G_{x_k}|} \int_{\{|\tilde{Z}| \leq \frac{\varepsilon}{4}\}} \psi_k(\tilde{Z}) e^{ip\theta_g} g^E(\tilde{Z}_{1,g}) \mathcal{L}im_u(\tilde{Z}_{1,g}) \mathcal{E}_{g, \tilde{Z}_{1,g}}(u, \sqrt{p}\tilde{Z}_{2,g}) dv_{\widetilde{TM}}(\tilde{Z}) \\ &\quad + \frac{\text{rk}(E)}{(2\pi)^n} \int_M \psi_k(x) \frac{\det(\dot{R}_x^L) \text{Tr}_{\Lambda^{0,q}} [e^{u\omega_{d,x}}]}{\det(1 - \exp(-u\dot{R}_x^L))} dv_M(x) + o(1). \end{aligned} \quad (4.9)$$

However, for  $g \neq 1$ , observe that

$$\begin{aligned} &\int_{\{|\tilde{Z}| \leq \frac{\varepsilon}{4}\}} \psi_k(\tilde{Z}) e^{ip\theta_g} g^E(\tilde{Z}_{1,g}) \mathcal{L}im_u(\tilde{Z}_{1,g}) \mathcal{E}_{g, \tilde{Z}_{1,g}}(u, \sqrt{p}\tilde{Z}_{2,g}) dv_{\widetilde{TM}}(\tilde{Z}) \\ &= p^{-\frac{\dim N_{x_k, g}}{2}} \int_{A(p, \varepsilon)} \psi_k\left(\left(\tilde{Z}_{1,g}, \frac{\tilde{Z}'_{2,g}}{\sqrt{p}}\right)\right) e^{ip\theta_g} g^E(\tilde{Z}_{1,g}) \mathcal{L}im_u(\tilde{Z}_{1,g}) \mathcal{E}_{g, \tilde{Z}_{1,g}}(u, \tilde{Z}'_{2,g}) dv_{\widetilde{TM}}(\tilde{Z}), \end{aligned} \quad (4.10)$$

where  $A(p, \varepsilon) = \{|\tilde{Z}_{1,g}|^2 + \frac{1}{p}|\tilde{Z}'_{2,g}|^2 \leq \frac{\varepsilon}{4}\}$ . As we have  $\langle (1 - g^{-1})\tilde{Z}'_{2,g}, \tilde{Z}'_{2,g} \rangle \geq c_1 |\tilde{Z}'_{2,g}|$  with  $c_1 > 0$ , we can see that  $\mathcal{E}_{g, \tilde{Z}_{1,g}}(u, \tilde{Z}'_{2,g})$  is exponentially decaying as  $|\tilde{Z}'_{2,g}| \rightarrow \infty$ . Thus, there is  $C > 0$  such that

$$\left| \int_{A(p, \varepsilon)} \psi_k\left(\left(\tilde{Z}_{1,g}, \frac{\tilde{Z}'_{2,g}}{\sqrt{p}}\right)\right) e^{ip\theta_g} g^E(\tilde{Z}_{1,g}) \mathcal{L}im_u(\tilde{Z}_{1,g}) \mathcal{E}_{g, \tilde{Z}_{1,g}}(u, \tilde{Z}'_{2,g}) dv_{\widetilde{TM}}(\tilde{Z}) \right| \leq C. \quad (4.11)$$

as a consequence, since  $\dim N_{x_k, g} > 0$  for  $g \neq 1$ , we deduce that all the integrals (4.10) are  $o(1)$  as  $p \rightarrow \infty$ . Thus, at the end, (4.9) turns out to reduce to

$$p^{-n} \int_M \psi_k(x) \operatorname{Tr}_{\Lambda^{0,q}}[e^{-\frac{u}{p}\square_p}(x, x)] dv_M(x) = \frac{\operatorname{rk}(E)}{(2\pi)^n} \int_M \psi_k(x) \frac{\det(\dot{R}_x^L) \operatorname{Tr}_{\Lambda^{0,q}}[e^{u\omega_{d,x}}]}{\det(1 - \exp(-u\dot{R}_x^L))} dv_M(x) + o(1). \quad (4.12)$$

From (4.7), (4.8) and (4.12), we find that for  $p \rightarrow \infty$ ,

$$p^{-n} \operatorname{Tr}_q[e^{-\frac{u}{p}\square_p}] = \frac{\operatorname{rk}(E)}{(2\pi)^n} \int_M \frac{\det(\dot{R}_x^L) \operatorname{Tr}_{\Lambda^{0,q}}[e^{u\omega_{d,x}}]}{\det(1 - \exp(-u\dot{R}_x^L))} dv_M(x) + o(1). \quad (4.13)$$

On the other hand, for  $x \in X$ , let  $\{\tilde{w}_j\}_{1 \leq j \leq n}$  be an local orthonormal frame of  $\widetilde{T^{1,0}M_{U_x}}$  such that  $\dot{R}^L \tilde{w}_j = a_j(\tilde{Z}) \tilde{w}_j$  and let  $\{\tilde{w}^j\}_{1 \leq j \leq n}$  be its dual basis. Then we have the following formula on  $\tilde{U}_x$ :

$$\begin{aligned} \omega_d(\tilde{Z}) &= - \sum_{j=0}^n a_j(\tilde{Z}) \tilde{w}^j \wedge i_{\tilde{w}_j}, \\ e^{u\omega_d(\tilde{Z})} &= \prod_{j=0}^n (1 + (e^{-ua_j(\tilde{Z})} - 1)) \tilde{w}^j \wedge i_{\tilde{w}_j}, \\ \frac{\det(\dot{R}_x^L) \operatorname{Tr}_{\Lambda^{0,q}}[e^{u\omega_{d,x}}]}{\det(1 - \exp(-u\dot{R}_x^L))} &= \left( \sum_{j_1 < \dots < j_q} e^{-u \sum_{k=1}^q a_{j_k}(x)} \right) \prod_{j=0}^n \frac{a_j(x)}{(1 - e^{-ua_j(x)})}. \end{aligned} \quad (4.14)$$

In particular, the term in the integral in the right-hand side of (4.13) is uniformly bounded for  $x \in M$  and  $u > 0$ , and moreover,

$$\lim_{u \rightarrow \infty} \frac{1}{(2\pi)^n} \frac{\det(\dot{R}_x^L) \operatorname{Tr}_{\Lambda^{0,q}}[e^{u\omega_{d,x}}]}{\det(1 - \exp(-u\dot{R}_x^L))} = (-1)^q \mathbf{1}_{M(q)}(x) \det(\dot{R}_x^L/2\pi), \quad (4.15)$$

where  $\mathbf{1}_{M(q)}$  denotes the indicator function of  $M(q)$ .

From Theorem 1.2 and (4.13), we have for  $0 \leq q \leq n$  and any  $u > 0$

$$\begin{aligned} \limsup_{p \rightarrow \infty} p^{-n} \sum_{j=0}^q (-1)^{q-j} \dim H^j(M, {}^p \otimes E) \\ \leq \frac{\operatorname{rk}(E)}{(2\pi)^n} \int_M \sum_{j=0}^q \frac{\det(\dot{R}_x^L) \operatorname{Tr}_{\Lambda^{0,q}}[e^{u\omega_{d,x}}]}{\det(1 - \exp(-u\dot{R}_x^L))} dv_M(x). \end{aligned} \quad (4.16)$$

This, together with (4.15) and dominated convergence for  $u \rightarrow \infty$ , gives

$$\limsup_{p \rightarrow \infty} p^{-n} \sum_{j=0}^q (-1)^{q-j} \dim H^j(M, {}^p \otimes E) \leq (-1)^q \operatorname{rk}(E) \int_{M(\leq q)} \det(\dot{R}_x^L/2\pi) dv_M(x). \quad (4.17)$$

Finally, we have

$$\det(\dot{R}_x^L/2\pi) dv_M(x) = \frac{1}{n!} \left( \frac{\sqrt{-1}}{2\pi} R^L \right)^n, \quad (4.18)$$

which conclude the proof of Theorem 1.1.  $\square$



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